MOAA 2022: Team Round

October 8th, 2022

Rules

- No mathematical texts, notes, or online resources of any kind are permitted. Rely on your brain and those of your teammates!
- Compasses, protractors, rulers, straightedges, graph paper, blank scratch paper, and writing implements are generally permitted, so long as they are not designed to give an unfair advantage.
- No computational aids (including but not limited to calculators, phones, calculator watches, and computer programs) are permitted on any portion of the MOAA.
- Individual may only receive help from members of their team. Consulting any other individual is grounds for disqualification.

How to Compete

- **In Person:** After completing the test, write your answers down in the provided Team Round answer sheet. The proctors will collect your answer sheets immediately after the test ends.
- **Online:** After completing the test, your team captain should input your answers, along with your Team ID and name, into the provided Team Round Google Form.

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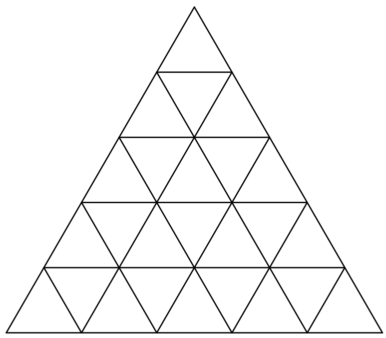
Team Round Problems

Welcome to the Team Round! The Team Round consists of 15 problems, ordered in approximately increasing difficulty, to be solved in 40 minutes. All answers are nonnegative integers no larger than 1,000,000.

T1. [**5**] Consider the 5 by 5 equilateral triangular grid as shown:



How many equilateral triangles are there with sides along the gridlines?

T2. [**5**] While doing her homework for a Momentum Learning class, Valencia draws two intersecting segments $AB = 10$ and $CD = 7$ on a plane. Across all possible configurations of those two segments, determine the maximum possible area of quadrilateral $ACBD$.

T3. [**7**] The area of the figure enclosed by the x-axis, y-axis, and line $7x + 8y = 15$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

T4. [**8**] Angeline flips three fair coins, and if there are any tails, she then flips all coins that landed tails each one more time. The probability that all coins are now heads can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

T5. [**10**] Find the smallest positive integer that is equal to the sum of the product of its digits and the sum of its digits.

T6. [**15**] Define a positive integer n to be *almost-cubic* if it becomes a perfect cube upon concatenating the digit 5. For example, 12 is *almost-cubic* because $125 = 5^3$. Find the remainder when the sum of all *almost-cubic* $n < 10^8$ is divided by 1000.

T7. [**20**] A point P is chosen uniformly at random in the interior of triangle ABC with side lengths $AB = 5$, $BC = 12$, $CA = 13$. The probability that a circle with radius $\frac{1}{3}$ centered at P does not intersect the perimeter of ABC can be written as $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

T8. [**20**] Raina the frog is playing a game in a circular pond with six lilypads around its perimeter numbered clockwise from 1 to 6 (so that pad 1 is adjacent to pad 6). She starts at pad 1, and when she is on pad i, she may jump to one of its two adjacent pads, or any pad labeled with j for which $j - i$ is even. How many jump sequences enable Raina to hop to each pad exactly once?

T9. [**25**] Emily has two cups A and B, each of which can hold 400 mL, A initially with 200 mL of water and B initially with 300 mL of water. During a *round*, she chooses the cup with more water (randomly picking if they have the same amount), drinks half of the water in the chosen cup, then pours the remaining half into the other cup and refills the chosen cup to back to half full. If Emily goes for 20 rounds, how much water does she drink, to the nearest integer?

T10. [**30**] Three integers A, B, C are written on a whiteboard. Every move, Mr. Doba can either subtract 1 from all numbers on the board, or choose two numbers on the board and subtract 1 from both of them whilst leaving the third untouched. For how many ordered triples (A, B, C) with $1 \leq A < B < C \leq 20$ is it possible for Mr. Doba to turn all three of the numbers on the board to 0?

T11. [**35**] Let a *triplet* be some set of three distinct pairwise parallel lines. 20 *triplets* are drawn on a plane. Find the maximum number of regions these 60 lines can divide the plane into.

T12. [**50**] Triangle ABC has circumcircle ω where B' is the point diametrically opposite B and C' is the point diametrically opposite C. Given $B'C'$ passes through the midpoint of AB, if $AC' = 3$ and $BC = 7$, find AB'^2.

T13. [**50**] Determine the number of distinct positive real solutions to

$$\lfloor x \rfloor^{\{x\}} = \frac{1}{2022}x^2.$$

Note: $\lfloor x \rfloor$ is known as the *floor* function, which returns the greatest integer less than or equal to x. Furthermore, $\{x\}$ is defined as $x - \lfloor x \rfloor$.

T14. [**50**] Find the greatest prime number p for which there exists a prime number q such that p divides $4^q + 1$ and q divides $4^p + 1$.

T15. [**70**] Let I_B, I_C be the B, C-excenters of triangle ABC, respectively. Let O be the circumcenter of ABC. If BI_B is perpendicular to AO, $AI_C = 3$ and $AC = 4\sqrt{2}$, then AB^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Note: In triangle $\triangle ABC$, the A-*excenter* is the intersection of the exterior angle bisectors of $\angle ABC$ and $\angle ACB$. The B-excenter and C-excenter are defined similarly.